

# MATHEMATICS

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(MAIN + ADVANCE) & COMPETITIVE EXAM.  
FOR XII (PQRS)**

## **BINARY OPERATIONS & Their Properties**

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### THINGS TO REMEMBER

1. A binary operation on a set  $S$  is a function from  $S \times S$  to  $S$ .  
A binary operation  $*$  on a set  $S$  associates any two elements  $a, b \in S$  to a unique element  $a * b \in S$ .
2. A binary operation  $*$  on a set  $S$  is said to be
  - (i) commutative, if  $a * b = b * a$  for all  $a, b \in S$ .
  - (ii) associative, if  $(a * b) * c = a * (b * c)$  for all  $a, b, c \in S$ .
  - (iii) distributive over a binary operation  $\circ$  on  $S$ , if
$$a * (b \circ c) = (a * b) \circ (a * c)$$
and  $(b \circ c) * a = (b * a) \circ (c * a)$  for all  $a, b \in S$
3. Let  $*$  be a binary operation on a set  $S$ . An element  $e \in S$  is said to be identity element for the binary operation  $*$ , if  $a * e = a = e * a$  for all  $a \in S$ .
4. Let  $*$  be a binary operation on a set  $S$  and  $e \in S$  be the identity element. An element  $a \in S$  is said to be invertible, if there exists an element  $b \in S$  such that
$$a * b = e = b * a$$
5. A binary operation on a finite set can be completely described by means of composition table. From the composition table, we can infer the following properties of the binary operation :
  - (i) The binary operation is commutative if the composition table is symmetric about the leading diagonal.
  - (ii) If the row headed by an element say  $e$  coincides with row at the top and the column headed by  $e$  coincides with the column on the extreme left, then  $e$  is the identity element.
  - (iii) If each row, except the top-most row, or each column, except the left-most column, contains the identity element. Then, every element of the set is invertible with respect to the given binary operation.
6. Total number of binary operations on a set consisting of  $n$  elements is  $n^{n^2}$ .

Total number of commutative binary operations on a set consisting of  $n$  elements is  $n^{\frac{n(n-1)}{2}}$ .

### EXERCISE-1

1. Let  $S$  be a non-empty set and  $P(S)$  be its power set. For any two subsets  $A$  and  $B$  of  $S$ , we know that  $A \cup B \subset S$ . That is, for any two elements of  $P(S)$ , we have  $A \cup B \in P(S)$ . Therefore, ' $\cup$ ' is a binary operation on  $P(S)$
2. Let  $S = \{a + \sqrt{2}b : a, b \in \mathbb{Z}\}$ . Then, prove that an operation  $*$  on  $S$  defined by
$$(a_1 + \sqrt{2}b_1) * (a_2 + \sqrt{2}b_2) = (a_1 + a_2) + \sqrt{2}(b_1 + b_2)$$
for all  $a_1, b_1, a_2, b_2 \in \mathbb{Z}$ .  
is binary operation on  $S$ .
3. Let  $M$  be the set of all singular matrices of the form  $\begin{bmatrix} x & x \\ x & x \end{bmatrix}$ , where  $x$  is a non-zero real number.  
On  $M$ , let  $*$  be an operation on  $M$ .
4. Is  $*$  defined on the set  $\{1, 2, 3, 4, 5\}$  by  $a * b = \text{LCM of } a \text{ and } b$  a binary operation ? Justify your answer.

5. Discuss the commutativity and associativity of the binary operation "\*" on  $\mathbb{R}$  defined by  $a * b = a + b + ab$  for all  $a, b \in \mathbb{R}$   
there on RHS we have usual addition, subtraction and multiplication of real numbers.
6. Discuss the commutativity and associativity of the binary operation \* on  $\mathbb{R}$  defined by  $a * b = \frac{ab}{4}$  for all  $a, b, \in \mathbb{R}$
7. Discuss the commutativity and associativity of binary operation "\*" defined on  $\mathbb{Q}$  by the rule  $a * b = a - b + ab$  for all  $a, b \in \mathbb{Q}$ .
8. Let  $A$  be a non-empty set and  $S$  be the set of all functions from  $A$  to itself. Prove that the composition of functions 'o' is a non-commutative binary operation on  $S$ . Also, prove that 'o' is an associative binary operation on  $S$ .
9. Let  $A = \mathbb{N} \times \mathbb{N}$  and "\*" be a binary operation on  $A$  defined by  $(a, b) * (c, d) = (ac, bd)$  for all  $a, b, c, d \in \mathbb{N}$ .  
Show that "\*" is commutative and associative binary operation on  $A$ .
10. Let  $A$  be a set having more than one element. Let "\*" be a binary operation on  $A$  defined by  $a * b = a$  for all  $a, b \in A$ .  
Is "\*" commutative or associative on  $A$  ?
11. Let "\*" be a binary operation on  $\mathbb{N}$ , the set of natural numbers, defined by  $a * b = a^b$  for all  $a, b \in \mathbb{N}$ .  
Is "\*" associative or commutative on  $\mathbb{N}$  ?
12. Let "\*" be a binary operation on  $\mathbb{N}$  given by  $a * b = \text{HCF}(a, b)$ ,  $a, b \in \mathbb{N}$   
(i) Find :  $12 * 4, 18 * 24, 7 * 5$   
(ii) Check the commutativity and associativity of "\*" on  $\mathbb{N}$ .
13. Consider the binary operations  $* : \mathbb{R} * \mathbb{R} \rightarrow \mathbb{R}$  and  $\circ : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  defined as  $a * b = |a - b|$  and  $a \circ b = a$  for all  $a, b \in \mathbb{R}$ .  
Show that \* is commutative but not associative,  $\circ$  is associative but not commutative. Further show that \* is distributive over  $\circ$ . Does  $\circ$  distribute over \* ? Justify your answer.
14. Determine which of the following binary operations are associative and which are commutative :  
(i) \* on  $\mathbb{N}$  defined by  $a * b = 1$  for all  $a, b \in \mathbb{N}$   
(ii) \* on  $\mathbb{Q}$  defined by  $a * b = \frac{a+b}{2}$  for all  $a, b \in \mathbb{Q}$ .
15. Let "\*" be a binary operation on a set  $S$ . If there exists an element  $e \in S$  such that  $a * e = a = e * a$  for all  $a \in S$ .  
Then,  $e$  is called an identity element for the binary operation "\*" on set  $S$ .
16. Let "\*" be a binary operation on a set  $S$ . If  $S$  has an identity element for "\*", then it is unique.
17. If \* is defined on the set  $\mathbb{R}$  of all real numbers by  $a * b = \sqrt{a^2 + b^2}$ , find the identity element in  $\mathbb{R}$  with respect to \*.
18. Let "\*" be an associative binary operation on a set  $S$  with the identity element  $e$  in  $S$ . Then, the inverse of an invertible element is unique.

19. Let  $*$  be an associative binary operation on a set  $S$  and  $a$  be an invertible element of  $S$ . Then,  
 $(a^{-1})^{-1} = a$

20. On  $Q$ , the set of all rational numbers, a binary operation  $*$  is defined by

$$a * b = \frac{ab}{5} \text{ for all } a, b \in Q.$$

21. Let  $'*'$  be a binary operation on  $Q_0$  (set of all non-zero rational numbers) defined by

$$a * b = \frac{ab}{4} \quad a, b \in Q_0.$$

Then, find the

- (i) Identity element in  $Q_0$
- (ii) inverse of an element in  $Q_0$ .

22. Let  $X$  be a non-empty set and let  $'*'$  be a binary operation on  $P(X)$  (the power set of  $X$ ) defined by  
 $A * B = A \cap B$  for  $A, B \in P(X)$

22. Let  $X$  be a non-empty set and let  $'*'$  be a binary operation on  $P(X)$  (the power set of set  $X$ ) defined by

$$A * B = (A - B) \cup (B - A) \text{ for all } A, B \in P(X)$$

23. Let  $A = Q \times Q$  and let  $*$  be a binary operation on  $A$  defined by

$$(a, b) * (c, d) = (ac, b + ad) \text{ for } (a, b), (c, d) \in A.$$

Then, with respect to  $*$  on  $A$

- (i) Find the identity element in  $A$
- (ii) Find the invertible elements of  $A$ .

24. Let  $A = N \cup \{0\} \times N \cup \{0\}$  and let  $'*'$  be a binary operation on  $A$  defined by

$$(a, b) * (c, d) = (a + c, b + d) \text{ for } (a, b), (c, d) \in A.$$

Show that :

- (i)  $'*'$  is commutative on  $A$ .
- (ii)  $'*'$  is associative on  $A$ .

25. Show that the number of binary operations on  $\{1, 2\}$  having 1 as identity and having 2 as inverse of 2 is exactly one.

26. Determine the total number of binary operations on the set  $S = \{1, 2\}$ . Then,  $*$  is a function from  $S \times S = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$  to  $S = \{1, 2\}$ .

27. Let  $'o'$  be a binary operation on the set  $Q_0$  of a non-zero rational numbers defined by

$$a o b = \frac{ab}{2}, \text{ for all } a, b, \in Q_0.$$

- (i) Show that  $'o'$  is both commutative and associate.
- (ii) Find the identity element in  $Q_0$ .
- (iii) Find the invertible elements of  $Q_0$ .

28. Let  $*$  be a binary operation on  $Z$  defined by

$$a * b = a + b - 4 \text{ for all } a, b, \in Z$$

- (i) Show that  $'o'$  is both commutative and associative.
- (ii) Find the identity element in  $Z$ .
- (iii) Find the invertible elements of  $Z$ .

29. Let  $*$  be the binary operation on  $N$  defined by  
 $a * b = \text{HCF of } a \text{ and } b.$   
 Does there exist identity for this binary operation on  $N$  ?
30. Let  $*$  be a binary operation on  $Q_0$  (set of non-zero rational numbers) defined by  
 $a * b = \frac{3ab}{5}$  for all  $a, b \in Q_0.$
31. Consider the set  $S = \{1, -1\}$  of square roots of unity and multiplication ( $\times$ ) as a binary operation on  $S$ . Construct the composition table for multiplication ( $\times$ ) on  $S$ . Also, find the identity element for multiplication on  $S$  and the inverses of various elements.
32. Consider the set  $S = \{1, -1, i, -i\}$  of fourth roots of unity. Construct the composition table for multiplication on  $S$  and deduce its various properties.
33. Consider the set  $S = \{1, 2, 3, 4\}$ . Define a binary operation  $*$  on  $S$  as follows :  
 $a * b = r$ , where  $r$  is the least non-negative remainder when  $ab$  is divided by 5.  
 Construct the composition table for  $*$  on  $S$ .
34. Construct the composition table for the composition of functions ( $\circ$ ) defined on the set  $S = \{f_1, f_2, f_3, f_4\}$  of four functions from  $C$  (the set of all complex numbers) to itself, defined by  
 $f_1(z) = z, f_2(z) = -z, f_3(z) = \frac{1}{z}, f_4(z) = -\frac{1}{z}$  for all  $z \in C.$
35. Consider the infimum binary operation  $\wedge$  on the set  $S = \{1, 2, 3, 4, 5\}$  defined by  
 $a \wedge b = \text{Minimum of } a \text{ and } b.$   
 Write the composition table of the operation  $\wedge$ .
36. Consider a binary operation  $*$  on the set  $\{1, 2, 3, 4, 5\}$  given by the following multiplication table
- |   |   |   |   |   |   |
|---|---|---|---|---|---|
| * | 1 | 2 | 3 | 4 | 5 |
| 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 2 | 1 | 2 | 1 |
| 3 | 1 | 1 | 3 | 1 | 1 |
| 4 | 1 | 2 | 1 | 4 | 1 |
| 5 | 1 | 1 | 1 | 1 | 5 |
- (i) Compute  $(2 * 3) * 4$  and  $2 * (3 * 4)$   
 (ii) Is  $*$  commutative ?  
 (iii) Compute  $(2 * 3) * (4 * 5)$
37. Define a binary operation  $*$  on the set  $A = \{0, 1, 2, 3, 4, 5\}$  as  
 $a * b = a + b \pmod{6}$   
 Show that zero is the identity for this operation and each element  $a$  of the set is invertible with  $6 - a$  being the inverse of  $a$ .
38. Define a binary operation  $*$  on the set  $\{0, 1, 2, 3, 4, 5\}$  as  

$$a * b = \begin{cases} a + b, & \text{if } a + b < 6 \\ a + b - 6, & \text{if } a + b \geq 6 \end{cases}$$
 Show that 0 is the identity for this operation and each element  $a \neq 0$  of the set is invertible with  $6 - a$  being the inverse of  $a$ .
39. Define a binary operation of 4.

40. Write the identity element for the binary operation  $*$  defined on the set  $R$  of all real numbers by the rule

$$a * b = \frac{3ab}{7} \text{ for all } a, b \in R$$

41. Let  $*$  be a binary operation, on the set of all non-zero real numbers, given by

$$a * b = \frac{ab}{5} \text{ for all } a, b \in R - \{0\}$$

42. Write the composition table for the binary operation multiplication modulo  $10$  ( $\times_{10}$ ) on the set  $S = \{2, 4, 6, 8\}$
43. Let  $*$  be a binary operation defined by  $a * b = 3a + 4b - 2$ . Find  $4 * 5$ .
44. Let  $*$  be a binary operation on  $N$  given by  $a * b = \text{HCF}(a, b)$ ,  $a, b \in N$ . Write the value of  $22 * 4$ .

### EXERCISE-3

1. An operation  $*$  is defined on the set  $Z$  of non-zero integers by  $a * b = \frac{a}{b}$  for all  $a, b \in Z$ . Then the property satisfied is

(a) closure (b) commutative (c) associative (d) None of these

2. Let  $*$  be a binary operation on  $Q^+$  defined by  $a * b = \frac{ab}{100}$  for all  $a, b \in Q^+$ . The inverse of  $0.1$  is

(a)  $10^5$  (b)  $10^4$  (c)  $10^6$  (d) non-existent

3. Consider the binary operation  $*$  defined on  $Q - \{1\}$  by the rule

$$a * b = a + b - ab \text{ for all } a, b \in Q - \{1\}$$

The identity element in  $Q - \{1\}$  is

(a) 0 (b) 1 (c)  $\frac{1}{2}$  (d) -1

4. For the multiplication of matrices as a binary operation on the set of all matrices of the form

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix}, a, b \in R \text{ the inverse of } \begin{bmatrix} 2 & 4 \\ -3 & 5 \end{bmatrix} \text{ is}$$

(a)  $\begin{bmatrix} -2 & 3 \\ -3 & -2 \end{bmatrix}$  (b)  $\begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$  (c)  $\begin{bmatrix} 2/13 & 3/13 \\ 3/13 & 2/13 \end{bmatrix}$  (d)  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

5. On the set  $Q^+$  of all positive rational numbers a binary operation  $*$  is defined by  $a * b = \frac{ab}{2}$  for all  $a, b \in Q^+$ . The inverse of  $8$  is

(a)  $\frac{1}{8}$  (b)  $\frac{1}{2}$  (c) 2 (d) 4

6. The number of binary operations that can be defined on a set of 2 elements is

(a) 8 (b) 4 (c) 16 (d) 64

7. For the binary operation  $*$  on  $Z$  defined by  $a * b = a + b + 1$  the identity element is

(a) 0 (b) -1 (c) 1 (d) 2